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ORIGINAL PAPER

Relations between edge lengths, dihedral and solid angles in tetrahedra

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Abstract The tetrahedron, fundamental in organic chemistry, is examined in view of two important kinds of angles: to each tetrahedron edge belongs a dihedral angle (internal intersection angle between two faces having the edge as common side) and to each tetrahedron vertex a solid angle (area of the surface inside the tetrahedron on the unit sphere with the vertex as center). Based on preliminary lemmas, these angles are expressed in terms of edge lengths by an essential use of determinants. The resulting formulae enable to specify angle properties by edge lengths, especially with regard to equality and inequality of single solid angles or certain sums of dihedral angles. A special kind of equal solid angles leads to symmetry aspects. Finally, it is shown that by a particular rearrangement of edges in tetrahedra of a specific class some derived angle properties will be preserved.

Keywords Tetrahedron · Dihedral angle · Solid angle · Determinant

Mathematics Subject Classification 52B15

André S. Dreiding, Professor Emeritus at the University of Zürich, deceased on December 24, 2013, at the age of 94. He was a pioneer in the field of mathematical chemistry. The Dreiding Stereomodels, invented in 1958 and used worlwide for around 40 years, are essentially characterized by geometrical ideas. Apart from his main activity in organic chemistry (i.a. elucidation of natural products, biosynthesic paths, totalsyntheses), André S. Dreiding started an interdisciplinary project in 1970 with the aim to systemize molecular model concepts in order to enable computer implementations (see for instance [3,9]). In his last years, as long as his health allowed it, he was involved in problems of mathematical chemistry [10–12].

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1 Introduction

We refer to a tetrahedron T as shown in Fig. 1 with the 4 vertices v_i and the 6 edges $v_i v_j$; for symmetry reasons it is $v_i v_j = v_j v_i$ $(1 \leq i, j \leq 4$ with $i \neq j)$. If i and j are two of the four vertex indices, the remaining two will always be named k and l $(k \neq l)$. To simplify the language, we use the notation e_{ij} for both the edges $v_i v_j$ as well as their lengths. Edges are said to be *adjacent* or *opposite* depending on whether they have a common vertex or not.

The dihedral and solid angles in T, the objects of our investigations, uniquely belong to the edges e_{ij} and vertices v_i, respectively, and they are defined as follows: A *dihedral angle* is the internal intersection angle of the two faces with common edge e_{ij}, written as α_{ij}. In analogy to the edges, we distinguish between adjacent and opposite dihedral angles. For the definition of a *solid angle* ϕ_i one considers the three adjacent edges e_{ij}, e_{ik} and e_{il} joining in v_i as well as the unit sphere with v_i as its center. The solid angle is then given by the area of the spherical triangle on the unit sphere where the vertices of the spherical triangle are the intersection points of the three edges with the sphere (of course, edges smaller than 1 must be extended beyond T). When we simply refer to angles we mean both dihedral and solid angles and it should be noted that all angles are given in radian measure.

Aspects of these angles will be algebraically expressed in terms of edges based on *sextuples*: Let $S = (e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34})$ be a sextuple of positive real numbers. If there exists a (non-degenerate) tetrahedron T such that the numbers of S are the edges of T, being arranged as indicated in Fig. 1, we say that *S determines T*. Note that the first three edges e_{12}, e_{13} and e_{14} are adjacent, while (e_{12}, e_{34}), (e_{13}, e_{24}) and (e_{14}, e_{23}) are pairs of opposite edges. Clearly, S describes a tetrahedron T up to isometry and there are 4! ways to determine a given T by sextuples which are mutually different if T is asymmetric.

2 Preliminary lemmas

Of course, three adjacent dihedral angles determine a solid angle. The fundamental relationship between these angles is based on the spherical excess. It was originally found by Hariot in 1603 (see for instance [7]). With our angle notations it can be written as follows:

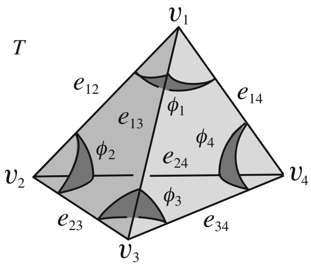

Fig. 1 Labeled tetrahedron T with solid angles

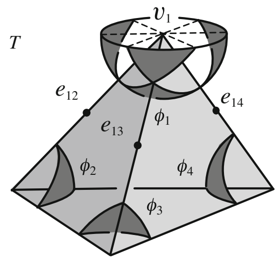

Fig. 2 Visualization of the proof of Lemma 2

Lemma 1 *In a tetrahedron T,*

$$\phi_i = \alpha_{ij} + \alpha_{ik} + \alpha_{il} - \pi. \tag{1}$$

Remark 1 A single dihedral angle in T is $<\pi$ and thus, by (1), a single solid angle becomes $<2\pi$, i.e., smaller than the expected half of the surface area of the unit sphere.

In the next two lemmas we give simple estimates of angle sums which will be used below. They can be found in literature on tetrahedra. In contrast to triangles with angle sum π, the sum of solid angles is not constant, but it holds:

Lemma 2 *In a tetrahedron T,*

$$\phi_1 + \phi_2 + \phi_3 + \phi_4 < 2\pi. \tag{2}$$

Proof In Fig. 2 the spherical triangles of the solid angles ϕ_2, ϕ_3 and ϕ_4 in T are reflected at the midpoints of the edges e_{12}, e_{13} and e_{14}, respectively. It can be seen that the sum of the solid angles is smaller than half of the surface area of the unit sphere with center v_1, i.e., $< 2\pi$. □

Remark 2 (a) The solid angle sum can assume all values of the interval $(0, 2\pi)$, since by moving v_1 the sum continuously varies and achieves the absolute extreme values 0 and 2π in cases of degenerate tetrahedra.
(b) From this and applying (1) to each summand of (2), it follows that the dihedral angle sum can assume all values of $(2\pi, 3\pi)$.

Two pairs of opposite edges (e_{ik}, e_{jl}) and (e_{il}, e_{jk}) form a skew quadrangle inside T. We denote this quadrangle by Q_{ij} or Q_{kl} because it is determined by the opposite vertices v_i and v_j or v_k and v_l, respectively.

Lemma 3 *For a quadrangle Q_{ij} in a tetrahedron T,*

$$\alpha_{ik} + \alpha_{il} + \alpha_{jl} + \alpha_{jk} < 2\pi. \tag{3}$$

Proof Consider to each of the four faces of T a normal vector ($\neq$ zero vector $\mathbf{0}$) pointing towards the interior of T. These vectors are linearly dependent and thus enable a linear combination which equals $\mathbf{0}$ as illustrated in Fig. 3 (each two successive

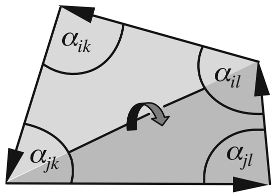

Fig. 3 Skew quadrangle with angles being the dihedral angles of Q_{ij} in Lemma 3

vectors are normal to faces which have an edge of Q_{ij} in common). We have a new skew quadrangle where the dihedral angles $\alpha_{ik}, \alpha_{il}, \alpha_{jl}$ and α_{jk} appear. It follows that their sum is smaller than 2π because by a rotation of one partial triangle around the diagonal we can obtain a planar quadrangle with the larger angle sum 2π. □

In the following, we want to take into consideration the edges of a tetrahedron. First, it should be emphasized that a sextuple $S = (e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34})$ must meet some conditions to ensure that it determines a tetrahedron T, a circumstance which was originally studied by Menger, Blumenthal, Herzog, Dekster, Wilker and others and was elaborated in a survey article [11]. Without proof we mention the main result. Calling a triple of the form (e_{ij}, e_{ik}, e_{jk}) with $e_{ij} < e_{ik} + e_{jk}$, $e_{ik} < e_{ij} + e_{jk}$ and $e_{jk} < e_{ij} + e_{ik}$ a *face triple* of S (fulfilled triangle inequality for one of the four faces of the potential T), the result can be formulated as follows:

Lemma 4 *A sextuple* $S = (e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34})$ *determines a tetrahedron* T *exactly if two conditions are fulfilled, namely* (i) S *has a face triple and* (ii) $D > 0$, *where* D *is the determinant of the matrix*

$$\mathbf{M} = \begin{pmatrix} 0 & e_{12}^2 & e_{13}^2 & e_{14}^2 & 1 \\ e_{12}^2 & 0 & e_{23}^2 & e_{24}^2 & 1 \\ e_{13}^2 & e_{23}^2 & 0 & e_{34}^2 & 1 \\ e_{14}^2 & e_{24}^2 & e_{34}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}. \tag{4}$$

Remark 3 (a) The two conditions (i) and (ii) imply that S has four face triples. This can be proved by using (9) below.

(b) The determinant D is usually called a *Cayley–Menger determinant*. It shows an immediate geometrical significance, namely

$$D = 288V^2, \tag{5}$$

where V denotes the volume of T. To prove this formula, V has to be expressed in terms of edges. The reader may perform this calculation which has already been done by the painter Piero della Francesca (∼1412–1492) and later by Euler (1758). From this formula (5) follows that the definition of D must be independent of the choice of the 4! possible sextuples S which determine T.

Cofactors of the matrix $\mathbf{M}$ in (4) will now be useful algebraic tools to examine the angles in a tetrahedron T. Consider the first 4 rows and columns of $\mathbf{M}$. For given i and j the term e_{ij}^2 appears twice, namely in the two rows and columns i and j. Referring to miners of submatrices obtained from $\mathbf{M}$ by not deleting the rows and columns i and j, we define the following cofactors (the indices specify the preserved rows and columns):

$$D_{ijk} := (l,l)\text{-minor of } \mathbf{M}, \quad D_{ij} := (-1)^{k+l} \cdot (k,l)\text{-minor of } \mathbf{M}. \tag{6}$$

Furthermore, we need a notation for the face areas of T:

$$\Delta_{ijk} := \text{area of the face } v_i v_j v_k. \tag{7}$$

Lemma 5 *The Cayley–Menger determinant D and the terms defined in (6) and (7) show the following properties:*

$$D_{ijk} = -16\Delta_{ijk}^2, \tag{8}$$

$$D_{ijk}\,D_{ijl} = 2e_{ij}^2\,D + D_{ij}^2, \tag{9}$$

$$D_{ijk} = D_{ijl} + D_{ikl} + D_{jkl} + 2(D_{il} + D_{jl} + D_{kl}). \tag{10}$$

Proof It follows from the factorization of the polynomial representation of D_{ijk} that the first property (8) is valid:

$$D_{ijk} = -(e_{ij} + e_{ik} + e_{jk})(e_{ij} + e_{ik} - e_{jk})(e_{jk} + e_{ij} - e_{ik})(e_{ik} + e_{jk} - e_{ij}).$$

Indeed, this term reveals that the stated formula is a consequence of Heron's theorem. Note that, due to the triangle inequality, D_{ijk} is always negative. The properties (9) and (10) can be verified by calculation. In a generalized form (9) appears by Blumenthal [1]. □

Remark 4 The cofactor D_{ij} has no geometrical significance. We still mention the polynomial representation of D_{ij} which will be used below:

$$D_{ij} = -e_{ij}^4 + (e_{ik}^2 + e_{il}^2 + e_{jk}^2 + e_{jl}^2 - 2e_{kl}^2)\,e_{ij}^2 + (e_{ik}^2 - e_{jk}^2)(e_{jl}^2 - e_{il}^2). \tag{11}$$

Note that $D_{ij} = D_{ji}$. This also follows from the symmetry of the matrix $\mathbf{M}$ in (4).

3 Computation of angles

In this section dihedral and solid angles in a tetrahedron will be expressed by its edges.

Theorem 1 *A dihedral angle α_{ij} in a tetrahedron T is given by*

$$\cos(\alpha_{ij}) = \frac{D_{ij}}{\sqrt{D_{ijk}D_{ijl}}}. \tag{12}$$

Proof The dihedral angle α_{ij} in T equals the intermediate angle between the two vectors $\overrightarrow{v_i v_j} \times \overrightarrow{v_i v_k}$ and $\overrightarrow{v_i v_j} \times \overrightarrow{v_i v_l}$ perpendicular on the two faces $v_i v_j v_k$ and $v_i v_j v_l$, respectively. In the following calculation we use Lagrange's identity, the cosine law as well as (8) and (11):

$$\begin{aligned}
\cos(\alpha_{ij}) &= \frac{(\overrightarrow{v_i v_j} \times \overrightarrow{v_i v_k}) \cdot (\overrightarrow{v_i v_j} \times \overrightarrow{v_i v_l})}{|\overrightarrow{v_i v_j} \times \overrightarrow{v_i v_k}| \;\; |\overrightarrow{v_i v_j} \times \overrightarrow{v_i v_l}|} \\
&= \frac{(\overrightarrow{v_i v_j} \cdot \overrightarrow{v_i v_j})\,(\overrightarrow{v_i v_k} \cdot \overrightarrow{v_i v_l}) - (\overrightarrow{v_i v_j} \cdot \overrightarrow{v_i v_k})\,(\overrightarrow{v_i v_j} \cdot \overrightarrow{v_i v_l})}{2\Delta_{ijk}\,2\Delta_{ijl}} \\
&= \frac{e_{ij}^2\,\frac{1}{2}(e_{ik}^2 + e_{il}^2 - e_{kl}^2) - \frac{1}{2}(e_{ij}^2 + e_{ik}^2 - e_{jk}^2)\,\frac{1}{2}(e_{ij}^2 + e_{il}^2 - e_{jl}^2)}{4\Delta_{ijk}\Delta_{ijl}} \\
&= \frac{-e_{ij}^4 + (e_{ik}^2 + e_{il}^2 + e_{jk}^2 + e_{jl}^2 - 2e_{kl}^2)\,e_{ij}^2 + (e_{ik}^2 - e_{jk}^2)(e_{jl}^2 - e_{il}^2)}{4\Delta_{ijk}\,4\Delta_{ijl}} \\
&= \frac{D_{ij}}{\sqrt{D_{ijk}D_{ijl}}}. \qquad \square
\end{aligned}$$

While the computation of dihedral angles is based on cosine, a convenient formula for the solid angles makes use of tangent. It is here derived based on formula (12) for dihedral angles but it could also be obtained more directly by applying a result given by Oosterom and Strackee [8]. In the formula for solid angles appears a further term N_i which is defined as follows:

$$N_i := (e_{ij} + e_{ik})(e_{ik} + e_{il})(e_{il} + e_{ij}) - (e_{ij}e_{kl}^2 + e_{ik}e_{jl}^2 + e_{il}e_{jk}^2). \tag{13}$$

Theorem 2 *A solid angle ϕ_i in a tetrahedron T is given by*

$$\tan(\tfrac{1}{2}\phi_i) = \frac{\sqrt{D/2}}{N_i}. \tag{14}$$

Proof We first compute $\sin(\alpha_{ij})$. Since $\alpha_{ij} < \pi$ and thus $\sin(\alpha_{ij}) > 0$, we obtain with (12), (9) and $D > 0$ (Lemma 4):

$$\sin(\alpha_{ij}) = \sqrt{1 - \cos^2(\alpha_{il})} = \frac{\sqrt{D_{ijk}D_{ijl} - D_{ij}^2}}{\sqrt{D_{ijk}D_{ijl}}} = \frac{e_{ij}\sqrt{2D}}{\sqrt{D_{ijk}D_{ijl}}}. \tag{15}$$

Now, we essentially use (1) and repeatedly apply the addition theorem for cosine:

$$\begin{aligned}
\cos(\phi_i) &= \cos(\alpha_{ij} + \alpha_{ik} + \alpha_{il} - \pi) \\
&= \cos(\alpha_{ij})\,\sin(\alpha_{ik})\,\sin(\alpha_{il}) + \sin(\alpha_{ij})\,\cos(\alpha_{ik})\,\sin(\alpha_{il}) + \\
&\quad \sin(\alpha_{ij})\,\sin(\alpha_{ik})\,\cos(\alpha_{il}) - \cos(\alpha_{ij})\,\cos(\alpha_{ik})\,\cos(\alpha_{il}).
\end{aligned}$$

Substituting the cosines and sines with the terms according to (12) and (15), respectively, it follows after some transformations:

$$\cos(\phi_i) = \frac{D_{ij} D_{ik} D_{il} - 2D\,(e_{ik}e_{il}D_{ij} + e_{il}e_{ij}D_{ik} + e_{ij}e_{ik}D_{il})}{D_{ijk}D_{ijl}D_{ikl}}. \tag{16}$$

Analogously, we obtain:

$$\sin(\phi_i) = \frac{\sqrt{2D}\,(e_{ij}D_{ik}D_{il} + e_{ik}D_{ij}D_{il} + e_{il}D_{ij}D_{ik} - 2e_{ij}e_{ik}e_{il}D)}{D_{ijk}D_{ijl}D_{ikl}}. \tag{17}$$

In order to receive an expression in tangent, we insert the terms of (16) and (17) in the trigonometric identity $\tan(\frac{1}{2}\phi_i) = (1 - \cos(\phi_i))/\sin(\phi_i)$. One verifies that the resulting fraction can be written in the form

$$\begin{aligned} \tan\left(\tfrac{1}{2}\phi_i\right) &= \frac{t_i\, D}{t_i\,\sqrt{2D}\,N_i}, \quad \text{where} \\ t_i &= (e_{ij} + e_{jk} - e_{ik})(e_{ij} + e_{jl} - e_{il})(e_{ik} + e_{jk} - e_{ij}) \\ &\quad (e_{ik} + e_{kl} - e_{il})(e_{il} + e_{jl} - e_{ij})(e_{il} + e_{kl} - e_{ik}). \end{aligned}$$

The stated formula (14) is obtained from reducing. □

4 Some properties of angles

Whether two dihedral angles are equal or not can be checked by computing them according to Theorem 1. In general, there is no other simple possibility based on edges to decide this. In the case of solid angles the situation is different, i.e., there exists a surprising criterion for their equality which has already been published in 2009 by Hajja [4]. Let p_i and p_j be the perimeters of the faces opposite the solid angles ϕ_i and ϕ_j, respectively, then we have:

Theorem 3 *For the equality of two solid angles in a tetrahedron T,*

$$\phi_i = \phi_j \;\Leftrightarrow\; p_i = p_j.$$

Proof From (2) follows that in the case of $\phi_i = \phi_j$ both solid angles in T must be smaller than π and thus $\frac{\pi}{2} > \phi_i/2 = \phi_j/2$. Since the tangent is biunique in $(0, \frac{\pi}{2})$ and by using (14) we can write:

$$\phi_i = \phi_j \;\Leftrightarrow\; \frac{\sqrt{D/2}}{N_i} = \frac{\sqrt{D/2}}{N_j} \;\Leftrightarrow\; N_i - N_j = 0.$$

Substituting N_i and N_j with the terms according to (13) and factorizing we equivalently obtain:

$$(e_{ik} + e_{il} - e_{jk} - e_{jl})(e_{ij} + e_{ik} + e_{jk})(e_{ij} + e_{il} + e_{jl}) = 0.$$

Thus, this zero-equation is fulfilled exactly if the quadrangle Q_{ij} satisfies

$$e_{jk} + e_{jl} = e_{ik} + e_{il}. \tag{18}$$

Adding e_{kl} on both sides completes the proof. □

Remark 5 Theorem 3 may be considered to be the 'pons asinorum' (asses bridge) for tetrahedra. This name was originally given to proposition 5 of Book 1 of Euclid's *Elements* which together with its converse in proposition 6 states that two angles of a triangle are equal exactly if the opposite sides are equal. 'Pons asinorum' has been probably used because proposition 5 was considered to be difficult for students to understand, the 'pons' thus 'bridging' the path to the more difficult propositions that follow.

Theorem 3 reveals: Whether two solid angles in a tetrahedron are equal or not depends, according to (18), only on four of the six edges. Consider the tetrahedron T of Fig. 1 and assume that $\phi_1 = \phi_2$ which is equivalent to $e_{23} + e_{24} = e_{13} + e_{14}$. Varying only the 'base edge' e_{12} or the opposite edge e_{34}, the solid angles ϕ_1 and ϕ_2 actually change but their equality will be preserved. This is the case if, for instance, one of the faces opposite ϕ_1 or ϕ_2 rotates arround e_{34} or one of the two other faces arround e_{12}.

Table 1 shows a tetrahedron with three equal solid angles which imply, by Theorem 3, equal perimeters but certainly not necessarily equal areas of opposite faces. In the example of Table 1 we additionally have: $\phi_1 > \phi_2 \Leftrightarrow p_1 > p_2$. The appropriate general result, also mentioned in a paper from 2012 by Hajja [5], looks as follows:

Corollary 1 *For the inequality of two solid angles in a tetrahedron T,*

$$\phi_i > \phi_j \Leftrightarrow p_i > p_j.$$

Proof From (2) follows that two cases are possible: $\pi/2 > \phi_i/2 > \phi_j/2$ and $\pi > \phi_i/2 > \pi/2 > \phi_j/2$. Since tangent is strictly monotone increasing and positive in $(0, \frac{\pi}{2})$, negative in $(\frac{\pi}{2}, \pi)$, the statement results by considering both cases of inequalities instead of equalities in the steps of the proof of Theorem 3. Note that the corollary can also be verified by considering continuous transformation of tetrahedron edges and applying Theorem 3. □

In the next theorem we present results about sums of adjacent and opposite dihedral angles:

Table 1 Illustration of Theorem 3 by an asymmetric tetrahedron T determined by $S = (5, 6, 7, 10, 9, 8)$

$\phi_1 = 1.945$	$\phi_2 = 0.215$	$\phi_3 = 0.215$	$\phi_4 = 0.215$
$p_1 = 27$	$p_2 = 21$	$p_3 = 21$	$p_4 = 21$
$\Delta_{234} = 34.197$	$\Delta_{134} = 20.333$	$\Delta_{124} = 17.412$	$\Delta_{123} = 11.399$

Theorem 4 *For a quadrangle Q_{ij} in a tetrahedron T,*

$$\alpha_{ik}+\alpha_{il}=\alpha_{jk}+\alpha_{jl} \Leftrightarrow e_{ik}+e_{il}=e_{jk}+e_{jl},$$
$$\alpha_{ik}+\alpha_{jl}=\alpha_{il}+\alpha_{jk} \Leftrightarrow e_{ik}+e_{jl}=e_{il}+e_{jk}.$$

Proof The first equivalence is an immediate consequence of (1) (common $\alpha_{ij}-\pi$ in the left-hand equation omitted) and (18):

$$\alpha_{ik}+\alpha_{il}=\alpha_{jk}+\alpha_{jl} \Leftrightarrow \phi_i=\phi_j \Leftrightarrow e_{ik}+e_{il}=e_{jk}+e_{jl}.$$

The proof of the second equivalence is more complicated because we cannot refer to equal solid angles in T. Applying the addition theorem for cosine and using (12) and (15) we obtain:

$$\cos(\alpha_{ik}+\alpha_{jl})=\frac{D_{ik}D_{jl}-2e_{ik}e_{jl}D}{\sqrt{D_{ijk}D_{ijl}D_{ikl}D_{jkl}}}. \tag{19}$$

From (3) follows that in our case of $\alpha_{ik}+\alpha_{jl}=\alpha_{il}+\alpha_{jk}$ both sums must be $<\pi$. With (19) and since the cosine is biunique in $(0,\pi)$ we find:

$$\alpha_{ik}+\alpha_{jl}=\alpha_{il}+\alpha_{jk} \Leftrightarrow \frac{D_{ik}D_{jl}-2e_{ik}e_{jl}D}{\sqrt{D_{ijk}D_{ijl}D_{ikl}D_{jkl}}}=\frac{D_{il}D_{jk}-2e_{il}e_{jk}D}{\sqrt{D_{ijk}D_{ijl}D_{ikl}D_{jkl}}}$$
$$\Leftrightarrow D_{ik}D_{jl}-D_{il}D_{jk}+2D\,(e_{il}e_{jk}-e_{ik}e_{jl})=0.$$

By calculation one verifies a factorization of the term in this zero-equation so that it equivalently looks as follows:

$$(e_{il}+e_{jk}-e_{ik}-e_{jl})(e_{il}+e_{jk}+e_{ik}+e_{jl})\,D=0.$$

Since $D>0$ this is equivalent to $e_{ik}+e_{jl}=e_{il}+e_{jk}$. □

As an example consider the tetrahedron in Table 2, where equalities according to Theorem 4 and addionally corresponding inequalites are indicated. The general statements about inequalities are as follows:

Table 2 Illustration of Theorem 4 and the subsequent Corollary 2 by an asymmetric tetrahedron T determined by $S=(2,4,6,3,7,9)$

$\alpha_{12}=2.111$	$\alpha_{13}=1.501$	$\alpha_{14}=0.897$	$\alpha_{23}=0.952$	$\alpha_{24}=1.446$	$\alpha_{34}=0.836$

$\alpha_{13}+\alpha_{14}=\alpha_{23}+\alpha_{24} \Leftrightarrow e_{13}+e_{14}=e_{23}+e_{24}$
$\alpha_{12}+\alpha_{34}=\alpha_{13}+\alpha_{24} \Leftrightarrow e_{12}+e_{34}=e_{13}+e_{24}$
$\alpha_{12}+\alpha_{14}>\alpha_{23}+\alpha_{34} \Leftrightarrow e_{12}+e_{14}<e_{23}+e_{34}$
$\alpha_{13}+\alpha_{24}>\alpha_{14}+\alpha_{23} \Leftrightarrow e_{13}+e_{24}>e_{14}+e_{23}$

Corollary 2 *For a quadrangle Q_{ij} in a tetrahedron T,*

$$\alpha_{ik} + \alpha_{il} > \alpha_{jk} + \alpha_{jl} \;\Leftrightarrow\; e_{ik} + e_{il} < e_{jk} + e_{jl},$$
$$\alpha_{ik} + \alpha_{jl} > \alpha_{il} + \alpha_{jk} \;\Leftrightarrow\; e_{ik} + e_{jl} > e_{il} + e_{jk}.$$

Proof The first equivalence is immediately implied by (1) and Corollary 1. The second equivalence can be verified as follows: From $\alpha_{ik} + \alpha_{jl} > \alpha_{il} + \alpha_{jk}$ and (3) we obtain $\alpha_{ik}+\alpha_{jl} \in I$ where $I = (\alpha_{il}+\alpha_{jk}, 2\pi - \alpha_{il} - \alpha_{jk})$. It can easily be seen that the cosine values of this interval I are equal and maximal at the endpoints. Thus we have: $\alpha_{ik} + \alpha_{jl} > \alpha_{il} + \alpha_{jk} \Leftrightarrow \cos(\alpha_{ik} + \alpha_{jl}) < \cos(\alpha_{il} + \alpha_{jk})$. With that it remains to consider equivalences of inequalities instead of equalities in the proof of Theorem 4. □

Apart from the 'pons asinorum', tetrahedra show further analogies to the triangle geometry. For example, there is a relationship which can be considered as the law of cosine for tetrahedra. It was already found in 1883 by Dostor [2] and can now be proved easily.

Theorem 5 *For a tetrahedron T,*

$$\Delta_{ijk}^2 = \Delta_{ijl}^2 + \Delta_{ikl}^2 + \Delta_{jkl}^2 - 2\big(\Delta_{ijl}\Delta_{ikl}\cos(\alpha_{il}) + \Delta_{ijl}\Delta_{jkl}\cos(\alpha_{jl}) + \Delta_{ikl}\Delta_{jkl}\cos(\alpha_{kl})\big).$$

Proof An equivalent transformation of (10) leads to:

$$-\frac{D_{ijk}}{16} = -\frac{D_{ijl}}{16} - \frac{D_{ikl}}{16} - \frac{D_{jkl}}{16} - 2\left(\frac{\sqrt{D_{ijl}D_{ikl}}}{16}\frac{D_{il}}{\sqrt{D_{ijl}D_{ikl}}} + \frac{\sqrt{D_{ijl}D_{jkl}}}{16}\frac{D_{jl}}{\sqrt{D_{ijl}D_{jkl}}} + \frac{\sqrt{D_{ikl}D_{jkl}}}{16}\frac{D_{kl}}{\sqrt{D_{ikl}D_{jkl}}}\right).$$

With (8) and (12) we obtain the stated formula. □

5 Angles and symmetry

Two spherical triangles which determine equal solid angles ϕ_i and ϕ_j in a tetrahedron are normally not isometric, but if they are, we speak of *isoequal* solid angles and write $\phi_i \equiv \phi_j$. Analogously, we write $p_i \equiv p_j$ if p_i and p_j are the perimeters of isometric faces. The following theorem also appears in Hajja [4], however, while his proof is purely algebraic we will work with a geometric approach.

Theorem 6 *For the isoequality of two solid angles in a tetrahedron T,*

$$\phi_i \equiv \phi_j \;\Leftrightarrow\; p_i \equiv p_j.$$

Proof We use a net of T, as shown in Fig. 4: The arcs (with radius 1) of the face angles σ_1, σ_2, σ_3 and τ_1, τ_2, τ_3 are isometric with the sides of the original spherical triangles determining the solid angles ϕ_i and ϕ_j, respectively. Due to the more general Theorem 3, it can be assumed throughout the proof condition (18).

The assumption $p_i \equiv p_j$ is satisfied exactly if

$$(e_{ik} = e_{jk} \wedge e_{il} = e_{jl}) \vee (e_{ik} = e_{jl} \wedge e_{il} = e_{jk}). \tag{20}$$

Since the condition within the first bracket leads to $\sigma_1 = \tau_1$, $\sigma_2 = \tau_2$, $\sigma_3 = \tau_3$ and the one within the the second bracket to $\sigma_1 = \tau_1$, $\sigma_2 = \tau_3$, $\sigma_3 = \tau_2$, we have in both cases isometric spherical triangles, i.e., $\phi_i \equiv \phi_j$.

Now, we prove the contrapositive of the converse: From (20) and (18) follows that $p_i \not\equiv p_j$ implies $(e_{ik} \neq e_{jk} \wedge e_{il} \neq e_{jl}) \wedge (e_{ik} \neq e_{jl} \wedge e_{il} \neq e_{jk})$. In order to verify $\phi_i \not\equiv \phi_j$ it suffices to show that each of the two face angles σ_2 and σ_3 differs from each of the two face angles τ_2 and τ_3: Since $(e_{ik} \neq e_{jk} \wedge e_{il} \neq e_{jl})$ we have $\sigma_2 \neq \tau_2$ and $\sigma_3 \neq \tau_3$. Further, assume $\sigma_2 = \tau_3$ or $\sigma_3 = \tau_2$; it would then follow from (18) that the bold quadrangle in Fig. 4 is a parallelogram, contrary to $(e_{ik} \neq e_{jl} \wedge e_{il} \neq e_{jk})$.□

Of course, isoequal solid angles are neccessary for the symmetry of a tetrahedron T but they are also sufficent. From the proof of Theorem 6 follows that $\phi_i \equiv \phi_j$ exactly if condition (20) is fulfilled and we have the following: Fig. 5a shows that $(e_{ik} = e_{jk} \wedge e_{il} = e_{jl})$ of (20) is equivalent to the existence of a mirror reflection (improper) symmetry with $\phi_i \equiv \phi_j$ and Fig. 5b that $(e_{ik} = e_{jl} \wedge e_{il} = e_{jk})$ of (20) is equivalent to the existence of a twofold rotation (proper) symmetry even with $\phi_i \equiv \phi_j$ and $\phi_k \equiv \phi_l$. We summarize:

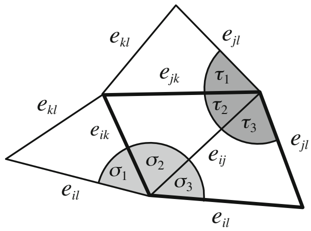

Fig. 4 Net of T as basis of the proof of Theorem 6

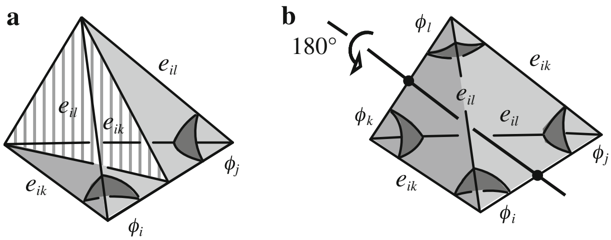

Fig. 5 Symmetric tetrahedra, in **a** with mirror and in **b** with twofold rotational symmetry

Corollary 3 *A tetrahedron T is symmetric exactly if it has (at least) two isoequal solid angles. A symmetric T has always as symmetry a mirror reflection or a twofold rotation.*

Remark 6 (a) The symmetries mirror reflection and twofold rotation generate, apart from the identity, possible further symmetries which are threefold rotations and fourfold rotation reflections.

(b) An achiral tetrahedron is always mirror-symmetric and a chiral tetrahedron has at most twofold rotation symmetries (since compositions of them are again twofold rotations). Of course, both would not necessarily be the case with any point sets.

(c) A necessary and sufficient condition for symmetry is not only given by isoequal solid angles but also by dihedral angles because,

$$(20) \;\Leftrightarrow\; (\alpha_{ik} = \alpha_{jk} \wedge \alpha_{il} = \alpha_{jl}) \;\vee\; (\alpha_{ik} = \alpha_{jl} \wedge \alpha_{il} = \alpha_{jk}).$$

The implication $\Rightarrow$ is evident due to the symmetries according to Fig. 5. The converse $\Leftarrow$ can be algebraically proved with help of Theorem 4. But it follows geometrically based on the fact that three adjacent angles uniquely determine the spherical triangle of a solid angle.

(d) It can easily be verified that two pairs of equal are two pairs of isoequal solid angles and thus yielding twofold rotational symmetry.

6 Angles in opposed tetrahedra

By rearranging the edges of a given tetrahedron T we can often obtain new anisometric tetrahedra. Usually, these tetrahedra have angle properties which are different from angle properties of T. But there is a class of tetrahedra which preserves some angle properties when the edges of T are rearranged in a particular way.

Consider to T a tetrahedron T' as shown in Fig. 6 where

$$e'_{ij} := e_{kl} \text{ for all } 1 \leq i < j \leq 4. \tag{21}$$

According to this definition, T' coincides with T in the opposite edges. One verifies: There is no further tetrahedron (i.e., anisometric to T and T') which has this property since, colloquially speaking, already one single 'exchange' of opposite edges in one of the tetrahedra T or T' leads to the other (an 'exchange' of the edges e_{12} and e_{34}, for instance, instead of three 'exchanges' as it is implied by Fig. 6). It follows that T and T' are isometric to each other if (at least) two opposite edges are equal.

In the case where any two opposite edges are unequal, T and T' are anisometric and we speak of *opposed* tetrahedra. These can be subdivided into two classes: *Class A* contains all tetrahedra where the three smallest of opposite edges are adjacent and *class* A' all tetrahedra where this is true for the three largest edges. Of course, Theorem 3 implies that tetrahedra of class A have exactly one largest and those of class A' exactly one smallest solid angle.

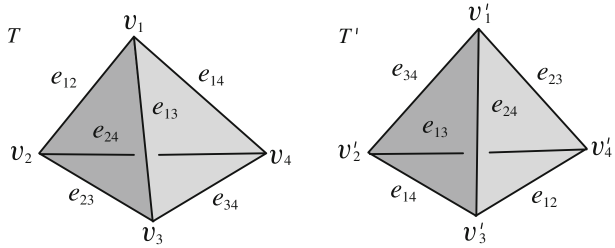

Fig. 6 Tetrahedra T and T' coinciding in the opposite edges

Tetrahedra of class A and rearrangment of edges which leads to opposed tetrahedra, i.e., tetrahedra of class A', is what now will be focused on. As a property of A, which can also be found in another context by Herzog [6], we have:

Theorem 7 *To any tetrahedron $T \in A$ there exists an opposed tetrahedron $T' \in A'$.*

Proof We use Lemma 4. Without loss of generality we can assume that T is determined by $S = (e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34})$ satisfying the A-class inequalities $e_{12} < e_{34}$, $e_{13} < e_{24}$ and $e_{14} < e_{23}$. It is to show that $S' = (e_{34}, e_{24}, e_{23}, e_{14}, e_{13}, e_{12})$ determines T'. First, we verify condition (i) of Lemma 4: Regarding three inequalities which are satisfied in T and applying the A-class inequalities we obtain

$$\begin{aligned} & e_{34} < e_{13} + e_{14} \ \wedge \ e_{24} < e_{12} + e_{14} \ \wedge \ e_{23} < e_{12} + e_{13} \\ & \Rightarrow e_{12} < e_{13} + e_{14} \ \wedge \ e_{13} < e_{12} + e_{14} \ \wedge \ e_{14} < e_{12} + e_{13}, \end{aligned}$$

so that S' has the face triple (e_{12}, e_{13}, e_{23}). Now, we verify condition (ii): Computing the difference of the Cayley–Menger determinants associated with S and S' and again using the A-class inequalities yields

$$D' - D = 2(e_{34}^2 - e_{12}^2)(e_{24}^2 - e_{13}^2)(e_{23}^2 - e_{14}^2) > 0,$$

and since $D > 0$ it follows $D' > 0$. □

Remark 7 (a) To a tetrahedron $T' \in A'$ there must not necessarily exist an opposed tetrahedron $T \in A$. As a counterexample we mention the right tetrahedron T' determined by the sextuple (2, 2, 2, 1, 1, 1).

(b) By the way, since $D' - D > 0$ it follows from (5) for the appropriate volumes $V < V'$ which confirms that T and T' are anisometric.

Referring to Fig. 6, we assume that $T \in A$ and define a bijection which assigns to each angle in T an angle in the opposed T':

$$\phi_i \mapsto \phi'_i \ \wedge \ \alpha_{ij} \mapsto \alpha'_{ij}. \tag{22}$$

The following theorem shows which angle properties of T become properties of the assigned angles in T', and conversely.

Table 3 Illustration of Theorem 8: (a) relations between the solid angles of the C_s-symmetric tetrahedron T determined by $S = (3, 4, 4, 6, 6, 5)$ and the opposed T', (b) relations between the sums of opposite dihedral angles of the asymmetric tetrahedron T determined by $S = (5, 4, 3, 7, 6, 6)$ and the opposed T'

(a)					
$\phi_1 = 2.411, p_1 = 17$	$\phi_2 = 0.214, p_2 = 13$	$\phi_3 = 0.214, p_3 = 13$	$\phi_4 = 0.214, p_4 = 13$		
$\phi'_1 = 0.181, p'_1 = 11$	$\phi'_2 = 0.673, p'_2 = 15$	$\phi'_3 = 0.673, p'_3 = 15$	$\phi'_4 = 0.673, p'_4 = 15$		
$\phi_1 > \phi_2 = \phi_3 \equiv \phi_4 \;\Leftrightarrow\; \phi'_1 < \phi'_2 = \phi'_3 \equiv \phi'_4$					
(b)					
$\alpha_{12} = 2.074$	$\alpha_{13} = 1.754$	$\alpha_{14} = 1.834$	$\alpha_{23} = 0.568$	$\alpha_{24} = 0.648$	$\alpha_{34} = 1.011$
$\alpha'_{12} = 1.587$	$\alpha'_{13} = 0.952$	$\alpha'_{14} = 0.764$	$\alpha'_{23} = 1.506$	$\alpha'_{24} = 1.318$	$\alpha'_{34} = 1.404$
$\alpha_{12} + \alpha_{34} > \alpha_{13} + \alpha_{24} = \alpha_{14} + \alpha_{23} \;\Leftrightarrow\; \alpha'_{12} + \alpha'_{34} > \alpha'_{13} + \alpha'_{24} = \alpha'_{14} + \alpha'_{23}$					

Theorem 8 *For the angles in opposed tetrahedra T and T',*

$$\phi_i > \phi_j \;\Leftrightarrow\; \phi'_i < \phi'_j, \quad \phi_i = \phi_j \;\Leftrightarrow\; \phi'_i = \phi'_j, \quad \phi_i \equiv \phi_j \;\Leftrightarrow\; \phi'_i \equiv \phi'_j,$$
$$\alpha_{ik} + \alpha_{jl} > \alpha_{il} + \alpha_{jk} \;\Leftrightarrow\; \alpha'_{ik} + \alpha'_{jl} > \alpha'_{il} + \alpha'_{jk},$$
$$\alpha_{ik} + \alpha_{jl} = \alpha_{il} + \alpha_{jk} \;\Leftrightarrow\; \alpha'_{ik} + \alpha'_{jl} = \alpha'_{il} + \alpha'_{jk}.$$

Proof Taking into account (21) and (22) we can conclude as follows: The equivalences of unequal and equal solid angles in T and T' are implied by Corollary 1 and Theorem 3, respectively, and by the additional fact that $p_i + p'_i$ and $p_j + p'_j$ equals the sum of all six edges. The equivalence of isoequal solid angles holds because $\phi_i \equiv \phi_j \Leftrightarrow (20) \Leftrightarrow \phi'_i \equiv \phi'_j$. Finally, the equivalences about sums of opposite dihedral angles immediately result from Corollary 2 and Theorem 4. □

Remark 8 (a) Expected equivalences concerning sums of adjacent angles are not mentioned because, by (1), they directly depend on those of solid angles, and conversely.
(b) Comparing two dihedral angles in T with the assigned two in T', there is no general relation with respect to equality or inequality. To show that the equality is not preseved, let us consider the tetrahedron T determined by $S = (3, 4, 4, 6, 5, 5)$ with equal adjacent dihedral angles $\alpha_{12} = \alpha_{23} = 1.322$ but unequal assigned dihedral angles $\alpha'_{12} = 0.744$ and $\alpha'_{23} = 1.829$.
(c) The reader may wish to verify that opposed tetrahedra T and T' coincide in the symmetries: They can both be asymmetric, C_s-symmetric or C_{3v}-symmetric.

Finally, in Table 3 we present examples to Theorem 8.

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